This file presents an issue of the two frame Rules GLUE-FRAME-$\boxed{\star}$ and GLUE-FRAME-$\boxed{\star}$ (repeated below), as defined in the main dissertation. We mentioned that the correctness criterion for glue rules does not hold as-is for the two frame rules. This file presents this proof. This is a proof by contradiction, and it was chosen to be removed from the main dissertation as this proof can be confusing. We first repeat the definition of the correctness criterion for glue rules, starting by the k correctness.

$$\textsc{glue-frame-}\boxed{\star}\quad \frac{\Phi, t \Downarrow \Phi'}{\phi \boxed{\star} \Phi, t \Downarrow \phi \boxed{\star} \Phi'}\; \mathit{itf}\left(\Phi\right) = \mathit{itf}\left(\Phi'\right) \qquad \textsc{glue-frame-}\boxed{\circ}\quad \frac{\Phi, t \Downarrow \Phi'}{M \boxed{\circ} \Phi, t \Downarrow M \boxed{\circ} \Phi'}\; \mathit{itf}\left(\Phi\right) = \mathit{itf}\left(\Phi'\right)$$

A semantic triple $\sigma^\sharp, t \Downarrow^\sharp r^\sharp$ is k correct, k being a number, if for any concrete derivation of depth less than k with conclusion $\sigma, t \Downarrow r$, then $\sigma \in \gamma\left(\sigma^\sharp\right)$ implies $r \in \gamma\left(r^\sharp\right)$. The criterion for glue rules proceeds as follow. The glue predicate *glue* is correct if for all k and each of its instance, the k correctness of all its premises implies the k correctness of the result:

$$\forall \left(\sigma_i^\sharp\right), \left(r_i^\sharp\right).\ \mathit{glue}\left(\left\{\left(\sigma_i^\sharp, r_i^\sharp\right)\right\}, \sigma^\sharp, r^\sharp\right) \implies \left(\forall i.\ \sigma_i^\sharp, t \Downarrow^\sharp r_i^\sharp \text{ is } k \text{ correct}\right) \implies \sigma^\sharp, t \Downarrow^\sharp r^\sharp \text{ is } k \text{ correct}$$

The definition of the k correctness does not mention the interaction with the different glue rules. Instead, the k correctness is a property of the concretisation of membraned formulae. The correctness theorem provides a way to prove that a glue rule is not correct: if the considered glue rule can be used in combination with a correct glue rule to produce an unsound semantic triple, then the considered glue rule is not correct and does not respect the glue criterion. We use this principle to prove that Rules GLUE-FRAME-$\boxed{\star}$ and GLUE-FRAME-$\boxed{\circ}$ are not correct in our model.

The dissertation has proven that Rule GLUE-WEAKEN (repeated below) is correct in Chapter 5, supposing that the order relation $\sqsubseteq$ is compatible with the concretisation γ (see Section 3.2.4 of the dissertation). Instead of taking the ordering $\preccurlyeq$ (which is indeed compatible with the concretisation function), we can take the weaker relation $\sqsubseteq$ defined below.

Definition 0.1. We define the maximal ordering of membraned formulae as the relation $\sqsubseteq$ such that for all Φ_1 and Φ_2, we have $\Phi_1 \sqsubseteq \Phi_2$ if and only if $\gamma\left(\Phi_1\right) \subseteq \gamma\left(\Phi_2\right)$.

$$\textsc{glue-weaken}\quad \frac{\Phi_1 \sqsubseteq \Phi_1' \qquad \Phi_1', t \Downarrow \Phi_2' \qquad \Phi_2' \sqsubseteq \Phi_2}{\Phi_1, t \Downarrow \Phi_2}$$

The maximal ordering is defined to minimally restrict what Rule GLUE-WEAKEN can do. In particular, it enables it do change membranes. We show below an unsound abstract derivation using Rules GLUE-FRAME-$\boxed{\star}$ and GLUE-WEAKEN (with the maximal ordering of Definition 0.1).

$$\dfrac{\dfrac{\dfrac{}{\left(- \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta, \eta\right), \mathit{skip} \Downarrow \left(- \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta, \eta\right)}\ \textsc{red-skip}}{\left(- \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta, \eta\right), \mathit{skip} \Downarrow \left(\eta_o \to \eta_i, l_1 \to l_2, \bullet \to l_1 \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta_i, \eta_i\right)}\ \textsc{glue-weaken}}{l_0 \mapsto \{\mathtt{g} : l_0\} \boxed{\star} \left(- \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta, \eta\right), \mathit{skip} \Downarrow l_0 \mapsto \{\mathtt{g} : l_0\} \boxed{\star} \left(\eta_o \to \eta_1, l_1 \to l_2, \bullet \to l_1 \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta_i, \eta_i\right)}\ \textsc{glue-frame-}\boxed{\star}$$

$$\begin{gathered} = \qquad\qquad\qquad\qquad = \\ \left(- \mid l_1 \mapsto \{\mathtt{g} : l_1,\ \mathtt{f} : l_1\}, \eta, \eta\right), \mathit{skip} \Downarrow \left(\eta_o \to \eta_i, l_1 \to l_2, \bullet \to l_1 \,\middle|\, l_2 \mapsto \{\mathtt{g} : l_2\} \boxed{\star} l_1 \mapsto \{\mathtt{f} : l_1\}, \eta_i, \eta_i\right) \end{gathered}$$

The maximal ordering enables us to change how the inner location l_1 is related to external locations: it starts related to the outer location l_1, but is then declared as an allocated location. Indeed, the two concretisations below are identical. Note that they use different valuations ρ^{ν} to build the same concrete states. Rule GLUE-WEAKEN thus accepts to change one to the other.

$$\gamma\left(\left(\eta_o \to \eta_i \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta_i, \eta_i\right)\right) = \gamma\left(\left(\eta_o \to \eta_i, l_1 \to l_2, \bullet \to l_1 \mid l_1 \mapsto \{\mathtt{f} : l_1\}, \eta_i, \eta_i\right)\right)$$

The result $\left(\eta_o \to \eta_i, l_1 \to l_2, \bullet \to l_1 \,\middle|\, l_2 \mapsto \{\mathtt{g} : l_2\} \boxed{\star} l_1 \mapsto \{\mathtt{f} : l_1\}, \eta_i, \eta_i\right)$ does not have an empty concretisation, but it does not relate to the concrete results built from corresponding initial semantic context. In particular, this derivation does not respect the statement of the correctness theorem. This theorem has been proven in Coq. By contradiction, we thus know that at least an hypothesis of this theorem is not respected. The only unproven hypothesis was that Rule GLUE-FRAME-$\boxed{\star}$ does respect the glue criterion, which is thus not the case.

Rule GLUE-FRAME-$\boxed{\circ}$ can be rejected as well with a similar derivation. We have proven that Rule GLUE-FRAME-$\boxed{\star}$ is not provable correct using the glue criterion. This does not mean that it is not correct. We firmly believe our formalism to be adaptable to prove their correctness.